North Sydney Boys High School

MATHEMATICS (EXTENSION 1)

2011 HSC Course Assessment Task 3
(Trial HSC Examination)

General instructions

- Working time – 2 hours (plus 5 minutes reading time)
- **Commence each new question on a new page**.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ○ 12M3A – Mr Lam
- ○ 12M3B – Mr Weiss
- ○ 12M3C – Mr Lin
- ○ 12M4A – Mr Fletcher/Mrs Collins
- ○ 12M4B – Mr Ireland
- ○ 12M4C – Mrs Collins/Mr Rezcallah

STUDENT NUMBER .... **# BOOKLETS USED**: ....

Marker's use only.

QUESTION	1	2	3	4	5	6	7	Total	%
MARKS	12	12	12	12	12	12	12	84	

Question 1 (12 Marks) Commence a NEW page. **Marks**

(a) Find $\displaystyle\int \frac{3x^2-2}{x^3-2x+1}\,dx$. **1**

(b) If $y=\sqrt{\cos^3 2x}$, find $\dfrac{dy}{dx}$ in simplest form. **3**

(c) Find $\displaystyle\int \frac{t}{\sqrt{1+t}}\,dt$ using the substitution $u=1+t$. **3**

(d) i. Show that $\dfrac{d}{dx}\left[\tan^{-1}\left(\dfrac{x}{2}\right)+\dfrac{2x}{x^2+4}\right]=\dfrac{16}{(x^2+4)^2}$ **3**

ii. Hence or otherwise, evaluate **2**

$$\int_{-2}^{2}\frac{dx}{(x^2+4)^2}$$

Leave your answer in exact form.

Question 2 (12 Marks) Commence a NEW page. **Marks**

(a) Solve $\dfrac{4}{2x-1}<1$ for x and sketch the solution on a number line. **3**

(b) If α, β and γ are the roots of $8x^3-6x^2+1=0$, evaluate

i. $\alpha+\beta+\gamma$ **1**

ii. $\alpha\beta+\beta\gamma+\alpha\gamma$ **1**

iii. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$ **1**

(c) Solve $\sin\theta-2\cos\theta=1$ for $0^\circ\leq\theta\leq 360^\circ$. **3**

(d) Find the second derivative of $y=\sin\left(x^3\right)$. Draw a sketch of the curve in the immediate neighbourhood of $x=0$. **3**

Question 3 (12 Marks) Commence a NEW page. **Marks**

(a) Prove that $\dfrac{\sin 2A}{\sin A} - \dfrac{\cos 2A}{\cos A} = \sec A$. **2**

(b) Show that $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$. **3**

(d) Find the values of a and b in the expression $f(x) = 2x^4 + ax^3 - 2x^2 + bx + 6$ so that it is exactly divisible by $2x+1$ and has a remainder of 12 when it is divided by $x-1$. **3**

(e) Sketch $y = \sin\left(\sin^{-1} x\right)$. **2**

Question 4 (12 Marks) Commence a NEW page. **Marks**

(a) Consider the curve $y = \dfrac{\log_e x}{x}$.

i. Show that this curve cuts the x axis at one point only. **1**

ii. Discuss the behaviour of $y = \dfrac{\log_e x}{x}$ as $x \to 0$. **1**

iii. Show the maximum value of y is $\dfrac{1}{e}$. **3**

iv. Sketch the curve. **1**

(b) A bowl of water, heated to 100°C is placed in a cool room maintained at −5°C. After t minutes, the temperature T of the water is changing so that

$$\frac{dT}{dt} = -k(T-B)$$

i. Show that $T = B + Ae^{-kt}$, where A and B are constants, satisfies this condition and find the values of A and B. **2**

ii. After 20 minutes in the cool room, the water is 40°C. How long after being put in the cool room until the water reaches 10°C? Answer to the nearest minute. **2**

(c) Sketch the function $y = \dfrac{1}{2}\sin^{-1}\left(\dfrac{x}{2}\right)$. **2**

Question 5 (12 Marks) Commence a NEW page. **Marks**

(a) Divide the interval $A(3,-2)$ and $B(-4,5)$ externally in the ratio $3:1$. **2**

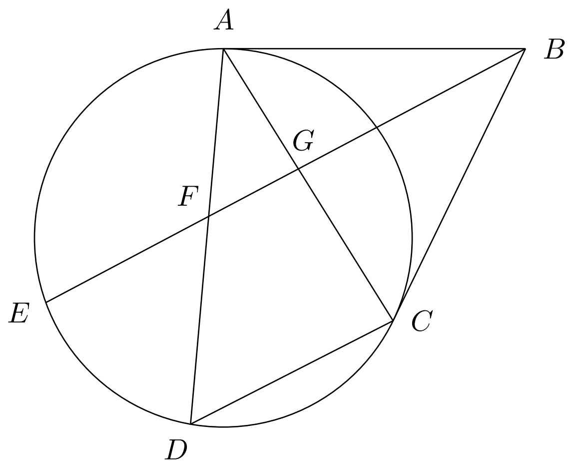
(b) Let $f(x) = 2x^3 + 2x - 1$.

i. Show that $f(x)$ has one root between $x = 0$ and $x = 1$. **1**

ii. By considering $f'(x)$, explain why there is only one root for $f(x)$. **1**

iii. Taking $x = 0$ as an initial approximation, use Newton's method to find a closer approximation. **2**

(c) In the diagram EB is parallel to DC. Tangents from B meet the circle at A and C. Copy the diagram into your writing booklet.



Prove that

i. $\angle BCA = \angle BFA$. **3**

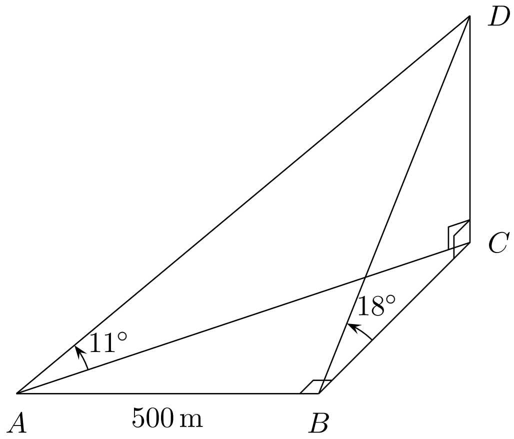
ii. $ABCF$ is a cyclic quadrilateral. **1**

iii. $DF = CF$. **2**

Question 6 (12 Marks) Commence a NEW page. **Marks**

(a) Prove using mathematical induction that $3^{2n+4} - 2^{2n}$ is divisible by 5. **3**

(b) The angle of elevation of the summit of a mountain due north is 18°. On walking 500 m due west the angle of elevation is found to be 11°. **3**



Find the height of the mountain.

(c) Prove that **2**

$$\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

where $\frac{d^2x}{dt^2}$ is the acceleration and v is the velocity.

(d) A particle is moving in a straight line and its acceleration is given by

$$\frac{d^2x}{dt^2} = \frac{1}{36 + x^2}$$

and is initially at rest at the origin.

i. Find v^2 as a function of x and explain why v is always positive for $t > 0$. **2**

ii. Find

A. The velocity at $x = 6$. **1**

B. The velocity as $t \to \infty$. **1**

Question 7 (12 Marks) Commence a NEW page. **Marks**

(a) i. Prove that if the displacement x of a particle P is related to the time t by the equation **1**

$$x = 3\cos(2\pi t)$$

then the motion is simple harmonic.

ii. Find the initial velocity. **1**

iii. Determine the greatest acceleration. **1**

iv. Determine when the particle is first at $x = \dfrac{3}{2}$. **1**

v. Express v^2 in terms of x and state the interval within which P is restricted. **2**

(b) i. Show that the equation of the normal at $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ is **2**

$$x + py = 2ap + ap^3$$

ii. The normal at P meets the y axis at N and M is the midpoint of PN. Find the coordinates of M. **2**

iii. Show that the locus of M is another parabola with its vertex being the focus of the original parabola. **2**

End of paper.

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \qquad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x,\ x > 0$

Suggested Solutions

Question 1 (Fletcher)

(a) (1 mark)

$$\int \frac{3x^2-2}{x^3-2x+1}\,dx = \log_e\left(x^3-2x+1\right)+C$$

(b) (3 marks)

- ✓ [1] for changing into index form.
- ✓ [1] for correct differentiation via chain rule.
- ✓ [1] for final answer in simplest form.

$$\begin{aligned} y &= (\cos 2x)^{\frac{3}{2}} \\ \frac{dy}{dx} &= \frac{3}{2}\times(-2\sin 2x)\times(\cos 2x)^{\frac{1}{2}} \\ &= -3\sin 2x\,(\cos 2x)^{\frac{1}{2}} \\ &= -3\sin 2x\sqrt{\cos 2x} \end{aligned}$$

(c) (3 marks)

- ✓ [1] for correctly changing variable of the integrand to u
- ✓ [1] for correct primitive in u.
- ✓ [1] for final answer in simplest form.

$$\int \frac{t}{\sqrt{1+t}}\,dt$$

Let $u=1+t$; then $du=dt$:

$$\begin{aligned} \int \frac{t}{\sqrt{1+t}}\,dt &= \int \frac{u-1}{u^{\frac{1}{2}}}\,du \\ &= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du \\ &= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\ &= \frac{2}{3}(1+t)^{\frac{3}{2}} - 2(1+t)^{\frac{1}{2}} + C \end{aligned}$$

(d) i. (3 marks)

- ✓ [1] for derivative of $\tan^{-1}\frac{x}{2}$.
- ✓ [1] for derivative of $\frac{2x}{x^2+4}$.
- ✓ [1] for final answer.

$$\begin{aligned} &\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{2}\right)+\frac{2x}{x^2+4}\right] \\ &= \frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2}{}^{\times 4}_{\times 4} + \frac{2x(x^2+4)-2x(2x)}{(x^2+4)^2} \\ &= \frac{2}{4+x^2}+\frac{8-2x^2}{(x^2+4)^2} \\ &= \frac{2(x^2+4)+8-2x^2}{(x^2+4)^2} \\ &= \frac{16}{(x^2+4)^2} \end{aligned}$$

ii. (2 marks)

- ✓ [1] for correct primitive.
- ✓ [1] for final answer.

$$\begin{aligned} &\int_{-2}^{2}\frac{dx}{(x^2+4)^2} \\ &= \frac{1}{16}\left[\tan^{-1}\frac{x}{2}+\frac{2x}{x^2+4}\right]_{-2}^{2} \\ &= \frac{1}{16}\left[\frac{\pi}{4}+\frac{1}{2}-\left(-\frac{\pi}{4}-\frac{1}{2}\right)\right] \\ &= \frac{1}{16}\left(\frac{\pi}{2}+1\right) \\ &= \frac{\pi+2}{32} \end{aligned}$$

Question 2 (Lin)

(a) (3 marks)

- ✓ [1] for multiplying both sides by square of denominator.
- ✓ [1] for obtaining correct roots for $(2x-1)(2x-5)$.
- ✓ [1] for final answer.

$$\begin{aligned} \underset{\times(2x-1)^2}{\frac{4}{2x-1}} &< \underset{\times(2x-1)^2}{1} \\ (2x-1)^2 &> 4(2x-1) \\ (2x-1)^2 - 4(2x-1) &> 0 \\ (2x-1)(2x-1-4) &> 0 \\ (2x-1)(2x-5) &> 0 \\ \therefore x < \frac{1}{2} &\text{ or } x > \frac{5}{2} \end{aligned}$$

(b) $8x^3 - 6x^2 + 0x + 1 = 0$.

i. (1 mark)

$$\alpha + \beta + \gamma = -\frac{b}{a} = \frac{3}{4}$$

ii. (1 mark)

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 0$$

iii. (1 mark)

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} \\ &= \frac{\frac{c}{a}}{-\frac{d}{a}} = -\frac{c}{d} = 0\end{aligned}$$

(c) (3 marks)

✓ [1] for expressing

$$\sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63^\circ 26')$$

✓ [1] for obtaining $\sin(\theta - 63^\circ 26') = \frac{1}{\sqrt{5}}$.

✓ [1] for final answers.

$$\begin{aligned}\sin\theta - 2\cos\theta &\equiv R\sin(\theta - \phi) \\ &= R\sin\theta\cos\phi - R\cos\theta\sin\phi\end{aligned}$$

Equating coefficients,

$$\begin{cases} R\cos\phi = 1 & (1) \\ R\sin\phi = 2 & (2) \end{cases}$$

(2) ÷ (1):

$$\begin{aligned}\tan\phi &= 2 \\ \therefore \phi &= 63^\circ 26'\end{aligned}$$

$(1)^2 + (2)^2$:

$$\begin{aligned}R^2\cos^2\phi + R^2\sin^2\phi &= 2^2 + 1^2 \\ \therefore R &= \sqrt{5} \\ \therefore \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63^\circ 26') &= 1 \\ \sin(\theta - 63^\circ 26') &= \frac{1}{\sqrt{5}} \\ \theta - 63^\circ 26' &= 26^\circ 34' \text{ or } 153^\circ 26' \\ \therefore \theta &= 90^\circ \text{ or } 216^\circ 52'\end{aligned}$$

(d) (3 marks)

$$\begin{aligned}y &= \sin\left(x^3\right) \\ \frac{dy}{dx} &= 3x^2\cos\left(x^3\right)\end{aligned}$$

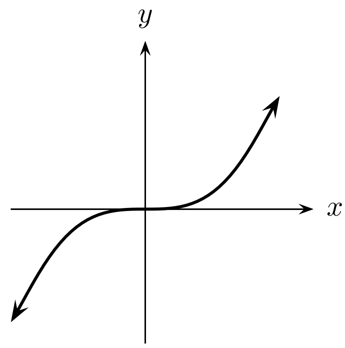
Apply product rule to find 2nd derivative,

$$\begin{aligned}u &= 3x^2 \quad v = \cos\left(x^3\right) \\ u' &= 6x \quad v' = -3x^2\sin\left(x^3\right) \\ \frac{d^2y}{dx^2} &= -9x^4\sin\left(x^3\right) + 6x\cos\left(x^3\right) \\ &= 3x\left(2\cos\left(x^3\right) - 3x^2\sin\left(x^3\right)\right) \\ \left.\frac{dy}{dx}\right|_{x=0} &= 0 \qquad \left.\frac{d^2y}{dx^2}\right|_{x=0} = 0\end{aligned}$$

From the information, ∃ possible horizontal point of inflexion around $x = 0$. Test sign of 2nd derivative:

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	$-$ ⌢	0	$+$ ⌣

Hence concavity changes around $x = 0$.



Question 3 (Weiss)

(a) (2 marks)

✓ [1] for obtaining $\frac{2\cos^2 A - \cos^2 A + \sin^2 A}{\cos A}$.

✓ [1] for final answer.

$$\begin{aligned}\frac{\sin 2A}{\sin A} - \frac{\cos 2A}{\cos A} &= \frac{2\cancel{\sin A}\cos A}{\cancel{\sin A}} - \frac{\cos^2 A - \sin^2 A}{\cos A} \\ &= \frac{2\cos^2 A - \cos^2 A + \sin^2 A}{\cos A} \\ &= \frac{\cos^2 A + \sin^2 A}{\cos A} = \sec A\end{aligned}$$

(b) (2 marks)

- ✓ [1] for transforming $\sin\theta$ and $\cos\theta$ to their t formulae.
- ✓ [1] for final answer.

$$\frac{1-\cos\theta}{\sin\theta} = \frac{1-\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{\frac{1+t^2-(1-t^2)}{\cancel{1+t^2}}}{\frac{2t}{\cancel{1+t^2}}}$$

$$= \frac{2t^2}{2t} = t = \tan\frac{\theta}{2}$$

(c) (3 marks)

- ✓ [1] for transforming $\cos^2\theta$ to $\frac{1}{2}+\frac{1}{2}\cos 2\theta$.
- ✓ [1] for evaluating primitive.
- ✓ [1] for final answer.

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta &= \int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta \\ &= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin\frac{\pi}{2} \\ &= \frac{\pi}{8} + \frac{1}{4} = \frac{\pi+2}{8}\end{aligned}$$

(d) (3 marks)

$$f(x) = 2x^4 + ax^3 - 2x^2 + bx + 6$$

By the remainder theorem, $f(1) = 12$:

$$\cancel{2} + a - \cancel{2} + b + \underset{-6}{6} = \underset{-6}{12}$$

$$a + b = 6$$

By the factor theorem, $f\left(-\frac{1}{2}\right) = 0$:

$$2\left(-\frac{1}{2}\right)^4 + a\left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right)^2 - \frac{1}{2}b + 6 = 0$$

$$\underbrace{\frac{1}{8} - \frac{a}{8} - \frac{1}{2} - \frac{b}{2}}_{\times 8} = \underset{\times 8}{-6}$$

$$1 - a - 4 - 4b = -48$$

$$a + 4b = 45$$

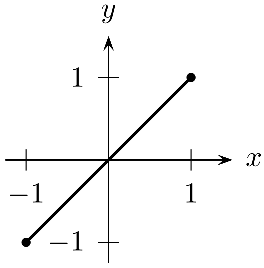
$$\begin{cases} a + b = 6 & (1) \\ a + 4b = 45 & (2) \end{cases}$$

(2) − (1):

$$\begin{aligned} 3b &= 39 \\ \therefore b &= 13 \\ \therefore a &= -7 \end{aligned}$$

(e) (2 marks)

- ✓ [1] for line.
- ✓ [1] for boundary points.



Question 4 (Lam)

(a) i. (1 mark)

$$y = \frac{\log_e x}{x} = 0$$

As $x \neq 0$,

$$\begin{aligned}\therefore \log_e x &= 0 \\ \therefore x &= 1\end{aligned}$$

ii. (1 mark)

$$\log_e x \ll x \text{ when } x < 1$$

$$\therefore \lim_{x\to 0} \frac{\log_e x}{x} = -\infty$$

iii. (3 marks)

- ✓ [1] for correctly applying quotient rule.
- ✓ [1] for $x = e$.
- ✓ [1] testing for local maximum.

$$u = \log_e x \qquad v = x$$

$$u' = \frac{1}{x} \qquad v' = 1$$

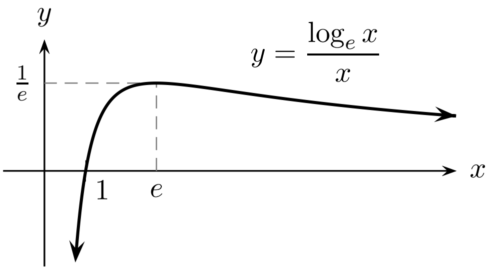
$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{x \times \frac{1}{x} - \log_e x}{x^2} \\ &= \frac{1 - \log_e x}{x^2}\end{aligned}$$

Stationary pts occur when $\frac{dy}{dx} = 0$,

$$
\begin{aligned}
1 - \log_e x &= 0 \\
\log_e x &= 1 \\
\therefore x &= e
\end{aligned}
$$

x		e	
$\frac{dy}{dx}$	$+$	0	$-$
y	↗	$\left(e, \frac{1}{e}\right)$	↘

iv. (1 mark)



(b) i. (2 marks)

✓ [1] for $A = 105$

✓ [1] for $B = -5$

$$
\begin{aligned}
\frac{dT}{dt} &= -k(T - B) \\
\frac{dT}{T - B} &= -k\,dt
\end{aligned}
$$

Integrating both sides,

$$
\begin{aligned}
\ln(T - B) &= -kt + C \\
T - B &= e^{-kt+C} = Ae^{-kt} \\
\therefore T &= B + Ae^{-kt}
\end{aligned}
$$

(Alternatively, differentiate and verify.)

By inspection,

$$B = -5$$

When $t = 0$, $T = 100°\text{C}$

$$
\begin{aligned}
100 &= -5 + Ae0 \\
\therefore A &= 105
\end{aligned}
$$

ii. (2 marks)

✓ [1] for finding the value of k.

✓ [1] for final answer.

When $t = 20$, $T = 40$

$$
\begin{aligned}
40 &= -5 + 105e^{-20k} \\
45 &= 105e^{-20k} \\
e^{-20k} &= \frac{3}{7} \\
-20k &= \log_e \frac{3}{7} \\
k &= -\frac{1}{20} \log_e \frac{3}{7} \\
\therefore T &= -5 + 105e^{\frac{1}{20} t \log_e \frac{3}{7}}
\end{aligned}
$$

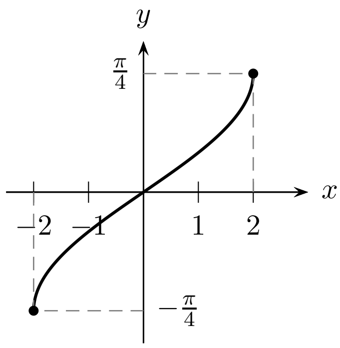
At $T = 10°\text{C}$,

$$
\begin{aligned}
10 &= -5 + 105e^{\frac{1}{20} t \log_e \frac{3}{7}} \\
15 &= 105e^{\frac{1}{20} t \log_e \frac{3}{7}} \\
e^{\frac{1}{20} t \log_e \frac{3}{7}} &= \frac{1}{7} \\
\frac{1}{20} t \log_e \frac{3}{7} &= -\log_e 7 \\
t &= \frac{-20 \log_e 7}{\log_e \frac{3}{7}} = 45.93\cdots = 46\,\text{min}
\end{aligned}
$$

(c) (2 marks)

✓ [1] for shape.

✓ [1] for endpoints and passing through origin.

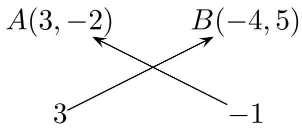


Question 5 (Collins)

(a) (2 marks)

✓ [1] for correct method.

✓ [1] for correct final answer.



$$\left(\frac{(3)(-1)+(-4)(3)}{3-1}, \frac{(-2)(-1)+(3)(5)}{2}\right)$$
$$= \left(\frac{-3-12}{2}, \frac{2+15}{2}\right) = \left(-\frac{15}{2}, \frac{17}{2}\right)$$

(b) i. (1 mark)

$$f(x) = 2x^3 + 2x - 1$$
$$f(0) = -1 \qquad f(1) = 3$$

As there is a change of sign in the y value between $x = 0$ and $x = 1$ and $f(x)$ is continuous between $x = 0$ and $x = 1$, therefore $f(x)$ has a root between $x = 0$ and $x = 1$.

ii. (1 mark)

✗ do not accept $f'(x) > 0$.

$$f'(x) = 6x^2 + 2$$

As $6x^2+2 > 0 \forall x$, therefore $f'(x)$ has no stationary points (therefore no turning points) at all. Hence there is only one root for $f(x)$.

iii. (2 marks)

✓ [1] for $f(0) = -1$, $f'(0) = 2$.

✓ [1] for final answer.

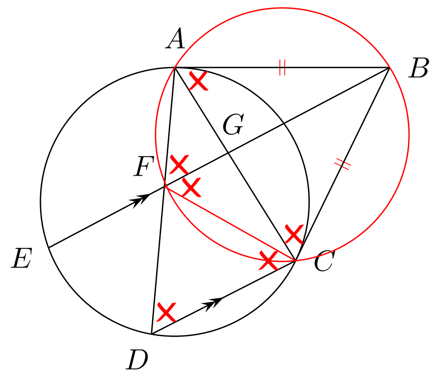
$$f(0) = -1 \qquad f'(0) = 2$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0 - \frac{-1}{2} = \frac{1}{2}$$
$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{8}\right) + \cancel{2\left(\frac{1}{2}\right)} - \cancel{1}$$
$$= \frac{1}{4}$$

(c) i. (3 marks)

✓ [1] for ∠ in isos △.

✓ [1] for ∠ alternate segment.

✓ [1] for corresponding ∠.



- $AB = CB$
  (tangents from external point)
- $\angle BCA = \angle BAC$
  (∠ opposite equal sides, isos $\triangle BCA$)
- $\angle BAC = \angle ADC$
  (∠ in alternate segment)
- $\angle ADC = \angle AFB$
  (corresponding ∠, $EB \parallel DC$)

$\therefore \angle BCA = \angle BFA$.

ii. (1 mark)
From the previous part, $\angle BCA = \angle BFA$. These angles represent ∠ at the circumference standing on arc AB. Hence C, F, A and B lie on the circumference of another circle, and $CFAB$ is a cyclic quadrilateral.

iii. (2 marks)

✓ [1] ONLY if incorrect working from (i) leads to proof that $\triangle DFC$ is isosceles.

✓ [1] ONLY if a property of cyclic quadrilaterals is correctly used. Otherwise,

✓ [1] for each dot point.

- $\angle CAB = \angle CFB$
  (∠ standing on the same arc CB)
- $\angle CFB = \angle FCD$
  (alternate ∠, $EB \parallel DC$)

$\therefore FD = FC$
(sides opposite equal ∠, isos $\triangle DFC$)

Question 6 (Ireland)

(a) (3 marks)

✓ [1] for testing base case.

✓ [2] for correct inductive step proof.

To prove $3^{2n+4} - 2^{2n}$ is divisible by 5 for all $n \geq 1$.

Base case: $n = 1$

$$3^{2(1)+4} - 2^{2(1)} = 3^6 - 2^2 = 725$$

which is divisible by 5. Hence the base case is true.

Inductive step: Assume $3^{2k+4} - 2^{2k}$ is divisible by 5, i.e.

$$\begin{aligned} 3^{2k+4} - 2^{2k} &= 5M \\ 3^{2k+4} &= 5M + 2^{2k} \end{aligned}$$

for k and $M \in \mathbb{Z}^+$. Examine the $(k+1)$-th term:

$$\begin{aligned} & 3^{2(k+1)+4} - 2^{2(k+1)} \\ &= 3^{2+2k+4} - 2^{2k+2} \\ &= 3^2 \cdot 3^{2(k+1)} - 2^2 \cdot 2^{2k} \\ &= 9\left(5M + 2^{2k}\right) - 4\left(2^{2k}\right) \\ &= 9 \cdot 5M + 9 \cdot 2^{2k} - 4 \cdot 2^{2k} \\ &= 9 \cdot 5M + 5 \cdot 2^{2k} \\ &= 5\left(9M + 2^{2k}\right) = 5P \end{aligned}$$

where $P(= 9M + 2^{2k}) \in \mathbb{Z}^+$.

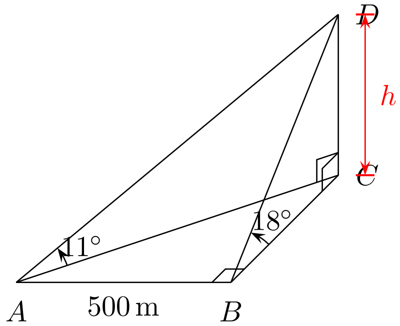
By the principle of mathematical induction, $3^{2n+4} - 2^{2n}$ is divisible by 5 for all positive integers n.

(b) (3 marks)

✓ [1] for BC and AC in terms of h.

✓ [1] for numerical expression in h^2.

✓ [1] for final answer.



In $\triangle BCD$

$$\frac{h}{BC} = \tan 18^\circ \quad \Rightarrow \quad BC = \frac{h}{\tan 18^\circ}$$

Likewise in $\triangle ACD$,

$$AC = \frac{h}{\tan 11^\circ}$$

As ABC is right angled, apply Pythagoras' Theorem:

$$\begin{aligned} AB^2 + BC^2 &= AC^2 \\ 500^2 + \left(\frac{h}{\tan 18^\circ}\right)^2 &= \left(\frac{h}{\tan 11^\circ}\right)^2 \\ 500^2 &= h^2\left(\frac{1}{\tan^2 11^\circ} - \frac{1}{\tan^2 18^\circ}\right) \\ \therefore h^2 &= \frac{500^2}{\frac{1}{\tan^2 11^\circ} - \frac{1}{\tan^2 18^\circ}} \\ &\approx \frac{500^2}{16.9943\cdots} \\ \therefore h &= 121.3\,\text{m (1 d.p.)} \end{aligned}$$

(c) (2 marks)

✓ [1] for obtaining $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dx}{dt} \cdot \frac{dv}{dx}$.

✓ [1] for correct proof.

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx} \quad \text{(chain rule)} \\ &= v\frac{dv}{dx} = \frac{dx}{dt} \cdot \frac{dv}{dx} \\ &= \frac{dv}{dt} = \frac{d^2x}{dt^2} = a \end{aligned}$$

(d) i. (2 marks)

✓ [1] for $v^2 = \frac{1}{3}\tan^{-1}\frac{x}{6}$.

✓ [1] for justification.

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{1}{36 + x^2} \\ \frac{1}{2}v^2 &= \int \frac{1}{36 + x^2}\,dx = \frac{1}{6}\tan^{-1}\frac{x}{6} + C \end{aligned}$$

When $t = 0$, $x = 0$ and $v = 0$:

$$\begin{aligned} \therefore 0 &= \frac{1}{6}(0) + C \\ \therefore C &= 0 \\ \therefore v^2 &= \frac{1}{3}\tan^{-1}\frac{x}{6} \end{aligned}$$

As $\frac{d^2x}{dt^2} > 0$ and velocity is initially 0, so for $t > 0$, the velocity will always be positive.

ii. A. (1 mark)
When $x = 6$,

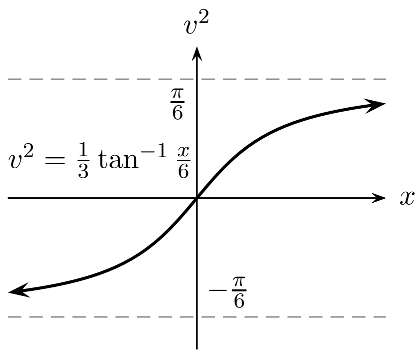
$$\begin{aligned} v^2 &= \frac{1}{3}\tan^{-1} 1 \\ &= \frac{1}{3} \times \frac{\pi}{4} = \frac{\pi}{12} \\ \therefore v &= \pm\sqrt{\frac{\pi}{12}} \end{aligned}$$

But from previous part, $v > 0$.

$$\therefore v = \sqrt{\frac{\pi}{12}}$$

B. (1 mark)
As $t \to \infty$, $x \to \infty$. Hence

$$\lim_{x\to\infty}\left(\frac{1}{3}\tan^{-1}\frac{x}{6}\right) = \frac{1}{3} \times \frac{\pi}{2} = \frac{\pi}{6}$$



$\therefore v \to \sqrt{\frac{\pi}{6}}$ as $t \to \infty$.

Question 7 (Collins)

(a) i. (1 mark)
✓ [1] only if 2nd derivative is correct and presented in SHM form.

$$\begin{aligned} x &= 3\cos(2\pi t) \\ \dot{x} &= -6\pi\sin(2\pi t) \\ \ddot{x} &= -12\pi^2\cos(2\pi t) \\ &= -4\pi^2 \times 3\cos(2\pi t) \\ &= -(2\pi)^2 \times 3\cos(2\pi t) \quad = -n^2 x \end{aligned}$$

As acceleration is proportional to displacement and directed to its opposite direction, therefore motion is SHM.

ii. (1 mark)
When $t = 0$, $\dot{x} = 0$.

iii. (1 mark)
Maximum acceleration when $\cos 2\pi t = -1$. $\therefore a_{\max} = 12\pi^2$ ms^{-2}

iv. (1 mark)
When $x = \frac{3}{2}$,

$$\begin{aligned} \frac{3}{2} &= 3\cos(2\pi t) \\ \cos(2\pi t) &= \frac{1}{2} \\ 2\pi t &= \frac{\pi}{3} \\ \therefore t &= \frac{1}{6} \end{aligned}$$

v. (2 marks)
✓ [1] for correct v^2.
✓ [1] for correct interval $x \in [-3, 3]$.

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4\pi^2 x$$

Integrating,

$$\frac{1}{2}v^2 = -4\pi^2 \times \frac{1}{2}x^2 = -2\pi^2 x^2 + C$$

When $t = 0$, $v = 0$ and $x = 3$.

$$
\begin{aligned}
0 &= -2\pi^2 \times (3^2) + C \\
\therefore C &= 18\pi^2 \\
\frac{1}{2}v^2 &= -2\pi^2 x^2 + 18\pi^2 \\
v^2 &= 36\pi^2 - 4\pi^2 x^2 \\
&= 4\pi^2\left(9 - x^2\right) \\
&\left(= n^2(a^2 - x^2)\right)
\end{aligned}
$$

Restriction on P:

$$
\begin{aligned}
9 - x^2 &\geq 0 \\
\therefore -3 \leq x &\leq 3
\end{aligned}
$$

(b) i. (2 marks)

✓ [1] for $m_\perp = -\frac{1}{p}$.

✓ [1] for final answer.

$$
\begin{aligned}
x^2 = 4ay \quad &\Rightarrow \quad y = \frac{x^2}{4a} \\
\frac{dy}{dx} = \frac{2x}{4a}\bigg|_{x=2ap} &= \frac{\cancel{4a}p}{\cancel{4a}} = p \\
\therefore m_\perp &= -\frac{1}{p}
\end{aligned}
$$

Using the point-gradient formula,

$$
\begin{aligned}
y - ap^2 &= -\frac{1}{p}(x - 2ap) \\
py - ap^3 &= -x + 2ap \\
\therefore x + py &= ap^3 + 2ap
\end{aligned}
$$

ii. (2 marks)

✓ [1] for coordinates of M.

✓ [1] for coordinates of N.

When the normal meets the y axis, $x = 0$.

$$
\begin{aligned}
0 + py &= ap^3 + 2ap \\
\therefore y &= ap^2 + 2a \\
\therefore N &\left(0, ap^2 + 2a\right)
\end{aligned}
$$

M is the midpoint of PN:

$$
\begin{aligned}
M &= \left(\frac{2ap + 0}{2}, \frac{ap^2 + (ap^2 + 2a)}{2}\right) \\
&= \left(ap, \frac{2ap^2 + 2a}{2}\right) = \left(ap, ap^2 + a\right)
\end{aligned}
$$

iii. (2 marks)

✓ [1] for locus.

✓ [1] for stating vertex of parabola.

$$
\begin{cases}
x = ap & (1) \\
y = a(p^2 + 1) & (2)
\end{cases}
$$

From (1),

$$
\therefore p = \frac{x}{a}
$$

Substitute to $y = a(p^2 + 1)$:

$$
\begin{aligned}
\therefore y &= a\left(\left(\frac{x}{a}\right)^2 + 1\right) \\
&= a\left(\frac{x^2}{a^2} + 1\right) \\
&= \frac{x^2}{a} + a \\
ay &= x^2 + a^2 \\
x^2 &= ay - a^2 = a(y - a)
\end{aligned}
$$

which is a parabola with vertex $(0, a)$ – the focus of the original parabola.